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On the Representation of the Remainder in the Variation-Diminishing Spline Approximation

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1. INTRODUCTION

Schoenberg [7] has defined a generalization of the Bernstein polynomials, by associating with $f(x)$, defined on [0, 1], the spline function of degree $k > 0$,

$$S_\Delta^k f(x) = \sum_{j=-k}^{n-1} f(\xi_j)\, N_j(x),$$

where $\Delta = \{x_i\}_{i=0}^n$ $(0 = x_0 < x_1 < \cdots < x_n = 1)$ and where the nodes ξ_j and the functions $N_j(x)$ depend on k and Δ. These functions are known as B-splines.

Recently Marsden [5] has extended the definition to generalized splines defined by means of extended Tchebycheff systems (see [2] and [3]); these splines will be referred to as Tchebycheffian B-splines. Marsden has obtained many results on uniform approximation of functions in $C[0, 1]$ using B-splines and Tchebycheffian B-splines; and also on the approximation properties of the corresponding derivatives. The last part of Marsden's paper is devoted to some conjectures of Voronovskaya type on the asymptotic behavior of the remainder. However, there are no results on the form of the remainder for fixed k and Δ. We shall represent the remainder by means of a generalized divided difference and estimate the rate of convergence associated with the B-splines to a given function.

We shall follow the notation of Marsden [5].

2. THE TCHEBYCHEFFIAN B-SPLINES

Let m be a positive integer and let $w_i(x)$ $(1 \leqslant i \leqslant m)$ be real-valued functions satisfying

$$w_i(x) \in C^{m+1}(-\infty, \infty) \qquad \text{and} \qquad \inf_{-\infty < x < \infty} w_i(x) > 0.$$

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Define the two systems $\{u_i(x)\}_{i=0}^{m-1}$ and $\{v_i(t)\}_{i=0}^{m+1}$ by

$$u_0(x) = 1,$$

$$u_1(x) = \int_0^x w_1(\xi_1)\, d\xi_1,$$

$$u_j(x) = \int_0^x w_1(\xi_1) \int_0^{\xi_1} w_2(\xi_2) \cdots \int_0^{\xi_{j-1}} w_j(\xi_j)\, d\xi_j \cdots d\xi_1, \qquad j = 2,..., m-1$$

and

$$v_0(t) = w_m(t),$$

$$v_1(t) = w_m(t) \int_0^t w_{m-1}(\xi_1)\, d\xi_1,$$

$$v_j(t) = w_m(t) \int_0^t w_{m-1}(\xi_1) \int_0^{\xi_1} w_{m-2}(\xi_2) \cdots \int_0^{\xi_{j-1}} w_{m-j}(\xi_j)\, d\xi_j \cdots d\xi_1, \qquad j = 2,..., m+1,$$

where $w_0(x) = w_{-1}(x) = 1$.

It is known that each of these systems is an extended complete Tchebycheff system (for detailed discussion see [2]).

Let $[a, b]$ be a finite interval, let $k = m - 1$ be positive and let $\Delta = \{x_1\}_{i=0}^n$ be a set of points in $[a, b]$ satisfying

$$\begin{gathered} a = x_0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1} < x_n = b, \\ x_{i-k} < x_i, \qquad k < i < n. \end{gathered} \tag{2.1}$$

Following Marsden [5] we extend this set by letting

$$x_j = a \quad j = -k,..., -1 \qquad \text{and} \qquad x_j = b \quad j = n+1,..., n+k.$$

Then let $N_j(x)$ $(-k \leqslant j \leqslant n-1)$ be the functions defined by Marsden [5] (8.4) and let ξ_j $(-k \leqslant j \leqslant n-1)$ be the nodes defined by [5] (9.2). It was proved by Marsden (see [5] (9.3)) that

$$a = \xi_{-k} < \xi_{-k+1} < \cdots < \xi_{n-1} = b, \tag{2.2}$$

$$\sum_{j=-k}^{n-1} N_j(x) = 1, \qquad a \leqslant x \leqslant b, \tag{2.3}$$

and

$$\sum_{j=-k}^{n-1} u_1(\xi_j)\, N_j(x) = u_1(x), \qquad a \leqslant x \leqslant b, \tag{2.4}$$

where $N_{-k}(a)$ is defined to be $N_{-k}(a+)$.

Further, with a function $f(x)$ defined on $[a, b]$ let us associate the Tchebycheffian B-spline

$$T_\Delta^k f(x) = \sum_{j=-k}^{n-1} f(\xi_j)\, N_j(x).$$

Then the following was proved by Marsden [5].

THEOREM M. *A sufficient condition that*

$$\lim T_\Delta^k f(x) = f(x) \quad \textit{uniformly in } [a, b]$$

for every $f \in C[a, b]$ *is that*

$$\lim k \,\| \Delta \| = 0, \tag{2.5}$$

where $\| \Delta \| = \max_{0 \leqslant i < n} (x_{i+1} - x_i)$.

For a function $f(x)$ defined on $[a, b]$, define the divided difference of order $\leqslant 2$ of $f(x)$, with respect to the function $u_1(x)$, by

$$f_{u_1}(t_0, t_1) = [f(t_0) - f(t_1)]/[u_1(t_0) - u_1(t_1)], \quad t_0 \neq t_1, \quad a \leqslant t_0, \quad t_1 \leqslant b,$$

and

$$f_{u_1}(t_0, t_1, t_2) = [f_{u_1}(t_0, t_1) - f_{u_1}(t_1, t_2)]/[u_1(t_0) - u_1(t_2)], \qquad t_0 \neq t_1 \neq t_2,$$
$$t_0 \neq t_2, a \leqslant t_0, t_1, t_2 \leqslant b.$$

The function $f(x)$ is called convex, non-concave, polynomial, nonconvex, or concave with respect to $u_1(x)$ if the divided difference of order 2 of $f(x)$ is positive, nonnegative, zero, non positive, or negative, respectively, for all triples t_0, t_1, t_2 of different points in $[a, b]$.

For the function $u_1(x) = x$, the above divided difference coincides with the ordinary divided difference and the definitions of convexity etc. coincide with the ordinary definitions.

3. MAIN RESULTS

Let $f(x)$ be defined in $[a, b]$ and denote the remainder in the approximation of $f(x)$ by $T_\Delta^k f(x)$, by

$$R_\Delta^k f(x) = f(x) - T_\Delta^k f(x). \tag{3.1}$$

Our main result is

THEOREM 1. *For fixed $k > 0$ and Δ satisfying* (2.1) *and any x, $a \leqslant x \leqslant b$, there are three different points in $[a, b]$ $\tau_i = \tau_i(k, \Delta, x)$, $i = 0, 2, 3$, such that for every continuous function $f \in C[a, b]$,*

$$R_\Delta^k f(x) = R_\Delta^k e_2(x) f_{u_1}(\tau_0, \tau_1, \tau_2), \tag{3.2}$$

where the function e_2 satisfies $e_2(x) = [u_1(x)]^2$.

Theorem 1 for the special case of the Bernstein polynomials was proved by Aramă [1] (see also [4]).

We shall need some preparatory results which are interesting by themselves. Let Δ' be the set of points in $[a, b]$ which is obtained from Δ by inserting a new partition point x' different from $x_0, \ldots, x_n$, and denote by $N_j'(x)$ and ξ_j' the corresponding $N_j(x)$'s and ξ_j's. We first prove

THEOREM 2. *Let $f(x)$ be defined on $[a, b]$ and let Δ and Δ' be two sets of points in $[a, b]$ with Δ satisfying* (2.1) *and Δ' is obtained from Δ by inserting x', $x_r < x' < x_{r+1}$. Then*

$$T_\Delta^k f(x) - T_{\Delta'}^k f(x) = \sum_{j=-k+r+1}^{r} a_j^2 f_{u_1}(\xi_{j-1}, \xi_j', \xi_j)\, N_j'(x), \tag{3.3}$$

where $a_j^2 = [u_1(\xi_j) - u_1(\xi_j')][u_1(\xi_j') - u_1(\xi_{j-1})] > 0 \quad -k + r + 1 \leqslant j \leqslant r$.

Proof. Let us start by expressing the $N_j(x)$'s by means of the $N_j'(x)$'s. Evidently

$$N_j(x) = N_j'(x), \qquad -k \leqslant j \leqslant -k + r - 1, \tag{3.4}$$

and

$$N_j(x) = N_{j+1}'(x), \qquad r + 1 \leqslant j \leqslant n - 1. \tag{3.5}$$

Now for $-k + r \leqslant j \leqslant r$, $N_j(x)$ is a Tchebycheffian spline of degree k over the knots of Δ' whose support lies in $[x_j, x_{j+k+1}]$. The functions $N_i'(x)$ form a basis for these splines (see [5], Theorem 5) and in fact it follows by [5] Theorem 5 that

$$N_j(x) = \alpha_j N_j'(x) - \beta_{j+1} N_{j+1}'(x) \qquad j = -k + r, \ldots, r \tag{3.6}$$

for some constants α_j and β_{j+1}.

In order to find α_j and β_{j+1} $j = -k + r, \ldots, r$ we employ the technique used by Marsden [5]. By (2.3) and (3.4) through (3.6)

$$\begin{aligned} 1 = \sum_{j=-k}^{-k+r-1} N_j'(x) + \alpha_{-k+r} N_{-k+r}'(x) + \sum_{j=-k+r+1}^{r} (\alpha_j - \beta_j)\, N_j'(x) \\ - \beta_{r+1} N_{r+1}'(x) + \sum_{j=r+2}^{n} N_j'(x). \end{aligned}$$

Hence by the unique representation of 1 by the $N_j'(x)$'s it follows that

$$\alpha_{-k+r} = 1, \quad \alpha_j - \beta_j = 1, \quad -k+r+1 \leqslant j \leqslant r, \quad \beta_{r+1} = -1. \tag{3.7}$$

Also by (2.4), (3.4)–(3.6) and (3.7),

$$\begin{aligned} u_1(x) &= \sum_{j=-k}^{-k+r-1} u_1(\xi_j)\, N_j'(x) + \sum_{j=-k+r}^{r} u_1(\xi_j)[\alpha_j N_j'(x) - \beta_{j+1} N_{j+1}'(x)] \\ &\quad + \sum_{j=r+1}^{n-1} u_1(\xi_j)\, N_{j+1}'(x) \\ &= \sum_{j=-k}^{-k+r} u_1(\xi_j)\, N_j'(x) + \sum_{j=-k+r+1}^{r} [\alpha_j u_1(\xi_j) - (\alpha_j - 1)\, u_1(\xi_{j-1})]\, N_j'(x) \\ &\quad + \sum_{j=r+1}^{n} u_1(\xi_{j-1})\, N_j'(x). \end{aligned}$$

Hence by the uniqueness of the representation of $u_1(x)$ by the $N_j'(x)$'s it follows that

$$\alpha_j u_1(\xi_j) - (\alpha_j - 1)\, u_1(\xi_{j-1}) = u_1(\xi_j'), \qquad -k+r+1 \leqslant j \leqslant r.$$

Therefore

$$\alpha_j = \frac{u_1(\xi_j') - u_1(\xi_{j-1})}{u_1(\xi_j) - u_1(\xi_{j-1})}, \qquad -k+r+1 \leqslant j \leqslant r. \tag{3.8}$$

In addition we see that

$$\xi_j = \xi_j', \quad -k \leqslant j \leqslant -k+r, \qquad \text{and} \quad \xi_j = \xi_{j+1}', \quad r \leqslant j \leqslant n-1, \tag{3.9}$$

which could also be proved directly.

By (3.7) and (3.8), (3.6) now takes the form

$$N_j(x) = \frac{u_1(\xi_j') - u_1(\xi_{j-1})}{u_1(\xi_j) - u_1(\xi_{j-1})} N_j'(x) + \frac{u_1(\xi_{j+1}) - u_1(\xi_{j+1}')}{u_1(\xi_{j+1}) - u_1(\xi_j)} N_{j+1}'(x), \tag{3.10}$$
$$-k+r \leqslant j \leqslant r.$$

Combining the above formulae, (3.3) is evident.

In order to complete the proof we need only show that

$$u_1(\xi_j) > u_1(\xi_j') \quad \text{and} \quad u_1(\xi_j') > u_1(\xi_{j-1}), \quad -k+r+1 \leqslant j \leqslant r. \tag{3.11}$$

Now it follows by the definition of the ξ_j's and the ξ_j''s that for $-k+r+1 \leqslant j \leqslant r$,

$$[u_1(\xi_j') - u_1(\xi_j)]\, N^* \begin{pmatrix} 0 & 1 & \cdots & m-2 \\ t & x_{j+1} & \cdots & x_{j+m-2} \end{pmatrix}$$

$$= N^* \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ t & x_{j+1} & \cdots & x_{j+m-1} \end{pmatrix}$$

$$- N^* \begin{pmatrix} 0 & 1 & \cdots & r-j, & r-j+1, & r-j+2 & \cdots & m-1 \\ t & x_{j+1} & \cdots & x_r & x' & x_{r+1} & \cdots & x_{j+m-2} \end{pmatrix}$$

(see [5] (8.1) and (8.5); also a similar result on top of p. 27 there). Letting t tend to x' we get

$$u_1(\xi_j') - u_1(\xi_j) = \frac{N^* \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ x' & x_{j+1} & \cdots & x_{j+m-1} \end{pmatrix}}{N^* \begin{pmatrix} 0 & 1 & \cdots & m-2 \\ x' & x_{j+1} & \cdots & x_{j+m-2} \end{pmatrix}} < 0.$$

The other part of (3.11) is proved similarly. This completes our proof.

The following is an immediate consequence (see also [5, Theorem 13] and the preceding lemma there).

COROLLARY 1. *Let Δ and Δ' be two sets of points in* $[a, b]$ *with Δ satisfying* (2.1) and *Δ' is obtained from Δ by inserting x', $a \leqslant x' \leqslant b$ different from the knots of Δ. Then if $f(x)$ is convex, nonconcave, polynomial, nonconvex, or concave with respect to $u_1(x)$, then $T_\Delta^k f(x)$ is greater, not less, equal, not greater, or less than $T_{\Delta'}^k f(x)$, respectively.*

Proof of Theorem 1. Let $k > 0$ and Δ satisfying (2.1) be given and construct a sequence Δ_m of sets of points in $[a, b]$ with the following properties. $\Delta_0 = \Delta$ and for $m \geqslant 1$, Δ_m is obtained from Δ_{m-1} by inserting an additional point $a \leqslant x_m' \leqslant b$ different from the knots of Δ_{m-1} in such a way that $\lim_{m\to\infty} \| \Delta_m \| = 0$. Obviously each Δ_m satisfies (2.1) and so by Corollary 1 if $f(x)$ is a convex function with respect to $u_1(x)$, then

$$T_{\Delta_m}^k f(x) > T_{\Delta_{m+1}}^k f(x), \qquad m = 0, 1, 2, \ldots.$$

Also by Theorem M, if $f(x)$ is continuous in $[a, b]$, then

$$\lim_{m\to\infty} T_{\Delta_m}^k f(x) = f(x).$$

Consequently if $f(x)$ is a continuous function in $[a, b]$, which is convex with respect to $u_1(x)$, then $T_{\Delta_m}^k f(x)$ decreases to $f(x)$ and thus

$$R_\Delta^k f(x) = f(x) - T_\Delta^k f(x) \neq 0,$$

for every continuous function $f(x)$, convex with respect to $u_1(x)$. Now our result follows by Popoviciu [6, Theorem 5]. In conclusion let us remark that $R_\Delta^k f(x)$ has the degree of exactness 1 (see [6, Section 25]) since by (2.3) and (2.4) $R_\Delta^k f(x)$ vanishes for the functions 1, $u_1(x)$.

The following is an immediate consequence of Theorem 1.

COROLLARY 2. *If the function* $g(x) = f(u_1^{-1}(x))$ *is twice continuously differentiable in* (a, b), *then for any fixed* $k > 0$ *and* Δ *satisfying* (2.1) *and any* x, $a \leqslant x \leqslant b$, *there exists* $\tau = \tau(k, \Delta, x)$ *such that*

$$R_\Delta^k f(x) = \tfrac{1}{2} R_\Delta^k e_2(x)\, g''(\tau).$$

When $u_1(x) = x$ the Tchebycheffian B-splines reduce to the B-splines that were introduced by Schoenberg [7]. For these splines we can establish an integral representation of the remainder. Applying Popoviciu [6, (84)] and the fact that $R_\Delta^k f(x)$ is of degree of exactness 1 we get

THEOREM 3. *For* $k > 0$ *and* Δ *satisfying* (2.1) *and every function* $f(x)$ *twice continuously differentiable in* $[a, b]$, *we have*

$$R_\Delta^k f(x) = f(x) - S_\Delta^k f(x) = \int_a^b R_\Delta^k \phi_t(x) f''(t)\, dt,$$

where

$$\phi_t(x) = (x - t + |x - t|)/2$$

and where $S_\Delta^k f(x)$ *denotes Schoenberg's B-spline (see* [7] *or* [5]).

4. Estimate of the Remainder for B-Splines

We give here some estimates of the remainder in the approximation of functions belonging to some subclasses of $C[a, b]$, using Schoenberg's B-splines. We use some estimates of $R_\Delta^k e_2(x)$ (here of course $e_2(x) = x^2$) due to Marsden [5]. Since, in this case $u_1(x) = x$, let us denote the ordinary divided difference of order 2 by $f(\tau_0, \tau_1, \tau_2)$.

THEOREM 4. *Let* $f \in C[a, b]$ *have a bounded ordinary divided difference of order* 2. *Then for* $k > 0$ *and* Δ *satisfying* (2.1),

$$|R_\Delta^k f(x)| \leqslant \min \left\{ \frac{(b-a)^2}{2k}, \frac{k \|\Delta\|^2}{2} \right\} \sup |f(\tau_0, \tau_1, \tau_2)|, \tag{4.1}$$

where the sup *is taken over all* τ_0, τ_1, τ_2 *three different points in* $[a, b]$

Proof. By [5] (4.1),

$$| R_\Delta^k e_2(x)| \leqslant \min \left\{ \frac{(b-a)^2}{2k}, \frac{k \| \Delta \|^2}{2} \right\}.$$

Our result follows now by Theorem 1.

COROLLARY 3. *If $f(x)$ is twice continuously differentiable in $[a, b]$, then for $k > 0$ and Δ satisfying* (2.1),

$$| R_\Delta^k f(x)| \leqslant \frac{1}{2} \max_{a \leqslant t \leqslant b} | f''(t)| \min \left\{ \frac{(b-a)^2}{2k}, \frac{k \| \Delta \|^2}{2} \right\}. \tag{4.2}$$

In view of the above results it seems that finding estimates of $R_\Delta^k e_2(x)$ would be useful in characterizing when $T_\Delta^k f(x)$ approximates $f(x)$. Marsden [5] (9.7), however, gives an estimate of $E(x) = T_\Delta^k u_2(x) - u_2(x)$ and uses this (see Theorem M in §2 here) to obtain a sufficient condition for $T_\Delta^k f(x)$ to approximate $f(x)$.

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